

CRITICAL PHENOMENA IN THE MAJORITY VOTER MODEL ON TWO DIMENSIONAL REGULAR LATTICES

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In this work we studied the critical behavior of the critical point as function of the number of nearest neighbors on two dimensional regular lattices. We performed numerical simulations on triangular, hexagonal and bilayer square lattices. Using standard finite size scaling theory we found that all cases fall in the two dimensional Ising model universality class, but that the critical point value for the bilayer lattice does not follow the regular tendency that the Ising model shows.

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I. INTRODUCTION

The Majority voter (MV) model is a simple non-equilibrium Ising-like system that presents an order-disorder phase transition in regular and complex lattices. Numerical results for the critical exponents, obtained on two-dimensional square lattices [1–3] and in three dimensional cubic lattice [4], are the same as those of the Ising model. Those results seem to confirm the conjecture that non equilibrium models with up-down symmetry and spin flip dynamics fall in the universality class of the equilibrium Ising model [5]. However, the critical exponents obtained in numerical simulations on non-regular lattices [3, 6–9], hypercubic lattices above four dimensions [10] and two dimensional regular lattices with honeycomb and triangular geometries [11] are different from the computed for the Ising model in the same geometries.

On the other hand the critical points reported for the MV model in Refs. [1] and [11] seem to indicated a non monotonic behavior as function of the number of nearest neighbors (z) in the lattice, the critical point for the square lattice ($z = 4$) is bigger than those of the triangular ($z = 6$) and the honeycomb ($z = 3$) lattices. This behavior clearly differs from the present in the Ising model for the same geometries, even if we include the result for the bilayer square lattice ($z = 5$) [12], where the inverse critical temperature β_c is a monotonic decreasing function on z .

We aim to clarify if the MV model belongs to the Ising model universality class in regular lattices and study the critical point dependence on the number of nearest neighbors in this model. In order to achieve our goals we eval-

uate the critical points and the critical exponents for two dimensional lattices in three different regular geometries: the honeycomb, the triangular and the bilayer square lattice.

II. MODEL AND FINITE SIZE SCALING

As mentioned above the MV model is an Ising-like system, in the sense that consists of a set of up-down "spins", each one located on a lattice site that interact, in this work, with its nearest neighbors. The system evolves in the following way: during an elementary time step, a spin $\sigma_i = \pm 1$ on the lattice is randomly picked up, and flipped with a probability given by

$$p(x) = \begin{cases} \frac{1}{2}(1+x) & \text{for } H_i \cdot \sigma_i < 0 \\ \frac{1}{2} & \text{for } H_i = 0 \\ \frac{1}{2}(1-x) & \text{for } H_i \cdot \sigma_i > 0 \end{cases}. \quad (1)$$

Here H_i is the local field produced by the nearest neighbors to the i th spin and x is the control parameter (noise). A given spin σ_i adopts the sign of H_i (majority) with probability $(1+x)/2$ and the opposite sign of H_i (minority) with probability $(1-x)/2$. In Refs. [1, 3, 6–9, 11] an equivalent evolution rule is used, in those works a given spin σ_i adopts the sign of the majority of its neighbors with probability p and the sign of the minority with probability $(1-p)$. It is clear that with both evolution rules the detailed balance condition is not satisfied.

The instantaneous order parameter m_t is defined as

$$m_t = \frac{1}{N} \sum_i \sigma_i, \quad (2)$$

where N is the total number of lattice sites. From here we can evaluate the moments of the order parameter as

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time averages

$$\langle m^k \rangle = \frac{1}{T - \tau} \sum_{t=\tau}^T |m_t|^k, \quad (3)$$

where τ is the transient time and $T - \tau$ is the running time. The susceptibility is given by

$$\chi = Nx\{\langle m^2 \rangle - \langle m \rangle^2\}. \quad (4)$$

We will use the method proposed in Ref. [13], where two different cumulants are used for the evaluation of the critical point, the fourth order-cumulant [14] (commonly known as Binder cumulant)

$$U^4 = 1 - \frac{\langle m^4 \rangle}{3\langle m^2 \rangle^2}, \quad (5)$$

and the second order cumulant

$$U^2 = 1 - \frac{2\langle m^2 \rangle}{\pi\langle m \rangle^2}. \quad (6)$$

We assume that the same scaling forms used in the equilibrium models can be applied for the MV model. So, we will have a free energy density given by the scaling ansatz

$$F(\epsilon, h, L) \approx L^{-(d-\alpha)/\nu} f^0(\epsilon L^{1/L}, hL^{(\beta+\gamma)/\nu}), \quad (7)$$

where $\epsilon = (x - x_c)$, x_c is the critical point for the infinite system, d is the dimension of the system, f^0 is a universal function, h is the symmetry-breaking (magnetic) field and L is the linear dimension ($N = L^2$ for the honeycomb and triangular lattices and $N = 2L^2$ for the two-layer lattice). The parameters α , β , γ and ν are the critical exponents for the infinite system. From (7) the scaling forms for the thermodynamic observables can be obtained, with $h = 0$ and one leading correction exponent, as

$$m(\epsilon, L) \approx L^{-\beta/\nu} (\hat{M}(\epsilon L^{1/\nu}) + L^{-\omega} \hat{\hat{M}}(\epsilon L)), \quad (8)$$

$$\chi(\epsilon, L) \approx L^{\gamma/\nu} (\hat{\chi}(\epsilon L^{1/\nu}) + L^{-\omega} \hat{\hat{\chi}}(\epsilon L)), \quad (9)$$

$$U^p(\epsilon, L) \approx \hat{U}^p(\epsilon L^{1/\nu}) + L^{-\omega} \hat{\hat{U}}^p(\epsilon L). \quad (10)$$

At the critical point $\epsilon = 0$ we obtain the following set of equations that allow us to evaluate the critical exponents for small lattice sizes:

$$m(L) \propto L^{-\beta/\nu} (1 + aL^{-\omega}), \quad (11)$$

$$\chi(L) \propto L^{\gamma/\nu} (1 + bL^{-\omega}), \quad (12)$$

and

$$\left. \frac{\partial U^p}{\partial x} \right|_{x=x_c} \propto L^{1/\nu} (1 + c_p L^{-\omega}), \quad (13)$$

here a , b and c_p are non universal constants. The critical point is evaluated taking in account that there are differences between the crossing points in U^2 for different

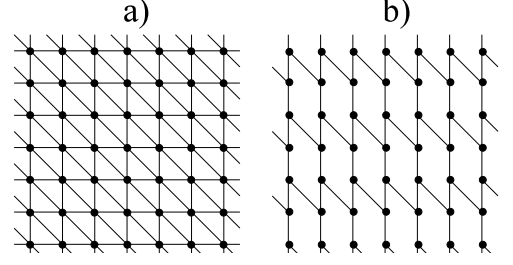


FIG. 1: a) Triangular and b) honeycomb lattices use in our simulations. Both geometries were tested on the Ising model.

values of L , with respect to the corresponding crossings evaluated for U^4 . The method take into account the correction-to-scaling effects on the crossing points. We expand Eq. (10) around $\epsilon = 0$ to obtain

$$U^p \approx U_\infty^p + \bar{U}^p \epsilon L^{1/\nu} + \bar{\bar{U}}^p L^{-\omega} + O(\epsilon^2, \epsilon L^{-\omega}), \quad (14)$$

where $p = 2$ or 4 and U_∞^p are universal quantities, but \bar{U}^p and $\bar{\bar{U}}^p$ are non-universal. The value of ϵ where the cumulant curves U^p for two different linear sizes L_i and L_j intercept is denoted as $\epsilon_{i,j}^p$. At this crossing point the following relation must be satisfied:

$$L_i^{1/\nu} \epsilon_{i,j}^p + B^p L_i^{-\omega} = L_j^{1/\nu} \epsilon_{i,j}^p + B^p L_j^{-\omega}. \quad (15)$$

Here $B^p = \bar{\bar{U}}^p / \bar{U}^p$. Combining for different cumulants ($q \neq p$) we get

$$\frac{x_{ij}^p + x_{ij}^q}{2} = x_c - (x_{ij}^p - x_{ij}^q) A_{pq}, \quad (16)$$

where $A_{pq} = (B^p + B^q) / [2(B^p - B^q)]$ and $x_{i,j}^p = \epsilon_{i,j}^p + x_c$. Equation (16) is a linear equation that makes no reference to ν or ω , and requires as inputs only the numerically measurable crossing couplings $x_{i,j}^p$. The intercept with the ordinate gives the critical point location. Additional details of the method can be found in Refs. [13] and [4].

III. RESULTS

We performed simulations on three different lattices with linear sizes $L = 24, 28, 32, 36, 40$ and 48 . For the triangular and honeycomb lattices we use the geometries shown in Figure 1 with periodic boundary conditions. For the bilayer square lattice we use a simple cubic lattice of size $N = 2 \times L \times L$ with periodic boundary conditions along the L direction and free boundary condition in the perpendicular direction, in this way we get a system whose critical behavior is two-dimensional (at least for the Ising model [12]) with $z = 5$. Starting with a random configuration of spins the system evolves following the dynamic rule explained in section II. In order to reach the stationary state we let the system evolves a transient

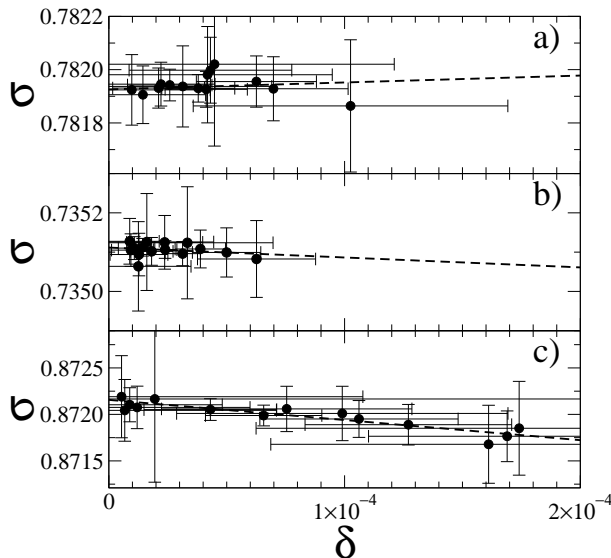


FIG. 2: Evaluation of the critical point for lattices with a) $z = 6$, b) $z = 5$ and c) $z = 3$ nearest neighbors. The circles are the numerical data obtained with third order polynomial fits and the dashed lines are the linear fits of Eq. (16). The smaller δ values correspond to the larger system sizes.

time that varied from 2×10^5 Monte Carlo time steps (MCTS) for $L = 24$ to 7×10^5 MCTS for $L = 48$. Averages of the observables were taken over 2×10^6 MCTS for $L = 24$ and up to 7×10^6 MCTS for $L = 48$. Additionally, for each value of x and L we performed up to 200 independent runs in order to improve the statistics. Our simulations were performed in the x ranges $[0.7785, 0.7850]$, $[0.734, 0.736]$ and $[0.869, 0.875]$ for the triangular, bilayer and honeycomb lattices respectively. For the evaluation of the critical points we use third order polynomial fitting for the cumulant curves. The estimation of the critical points for the three cases are shown in Figure 2, where we use the notation $\delta = x_{ij}^4 - x_{ij}^2$ and $\sigma = (x_{ij}^4 + x_{ij}^2)/2$.

The linear fits of Eq. (16) give the following estimated for the critical points $x_c = 0.7271(1)$, $0.7351(1)$ and $0.8721(1)$ for $z = 6$, 5 and 3 respectively. The uncertainties in our results are one order of magnitude smaller with respect to the values reported in Ref. [11]. We also note that the smaller values in δ are around 10^{-5} , as these values correspond to the crossings between the largest sizes using in our simulation we can be sure that largest sizes will not improve significantly our results.

Once that we have the critical points we can evaluate the critical exponents. For the evaluation of the critical exponent ν we use (13) with both cumulants U^2 and U^4 . In Figure 3 we are showing the derivatives of the cumulants at the critical point for the three lattices.

The results for $1/\nu$ from the power law fits are given in Table I. All results are in good agreement with the known value $1/\nu = 1$ of the two dimensional Ising model.

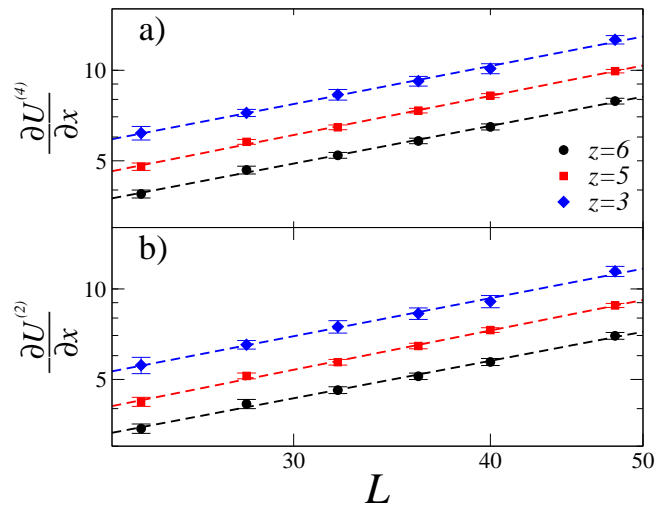


FIG. 3: (Color online) Log-log plot of the cumulant a) U^4 and b) U^2 derivatives at the critical point, for $z = 6$ (black circles), 5 (red squares) and 3 (blue diamonds). The dashed lines are power law fittings.

TABLE I: Estimates for $1/\nu$ obtained from the power law fitting of the cumulant derivatives at the critical point.

	$z = 6$	$z = 5$	$z = 3$
U^2	1.03(2)	1.04(5)	1.01(2)
U^4	1.03(2)	1.04(5)	1.01(2)

For the critical exponent γ we are using Eq. (12) to fit our data at the critical point (see Figure 4). Our results are $\gamma/\nu = 1.759(7)$, $1.756(9)$ and $1.755(8)$ for $z = 6$, 5 and 3 respectively. Again the agreement is acceptable compared with the value $\gamma/\nu = 7/4$ for the Ising model.

The fitting for the β exponents are shown in Figure 5. Our estimates are $\beta/\nu = 0.123(2)$, $0.123(3)$ and $0.123(2)$ for $z = 6$, 5 and 3 respectively. Those results also are in good agreement with $\beta/\nu = 1/8$ for the Ising model.

It is important to point out that correction to scaling, via the exponent ω , were not necessary in the evaluation of the critical exponents in the three geometries of this work. The same behavior was observed for the MV model in square lattices [1–3]. We summarized our results in Table II, along with the reported values in Ref. [11] for the MV model in honeycomb and triangular lattices and the Ising model in bilayer square lattice from Ref. [12]. We include also the results for the Rushbrooke-Josephson hyperscaling relation $d = (\gamma + 2\beta)/\nu$, which is satisfied by our results. The fact that relation is compatible with $d = 2$ by the bilayer square lattice for the MV and Ising models indicate that both are two dimensional systems.

The discrepancies between the results found in this work and the ones reported in Ref. [11] can be explained

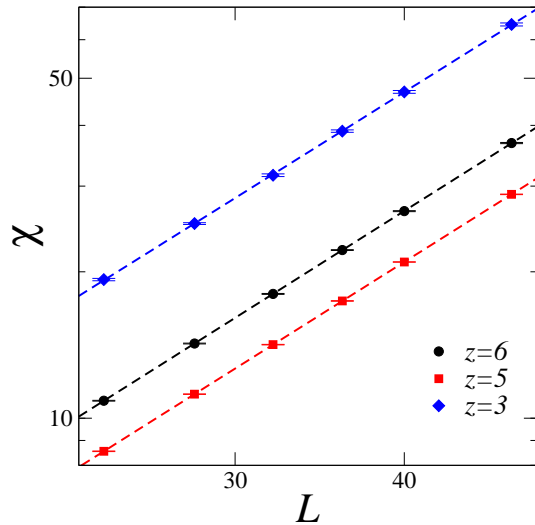


FIG. 4: (Color online) Log-log plot of the susceptibility at the critical point for $z = 6$ (black circles), 5 (red squares) and 3 (blue diamonds). The dashed line are power law fittings.

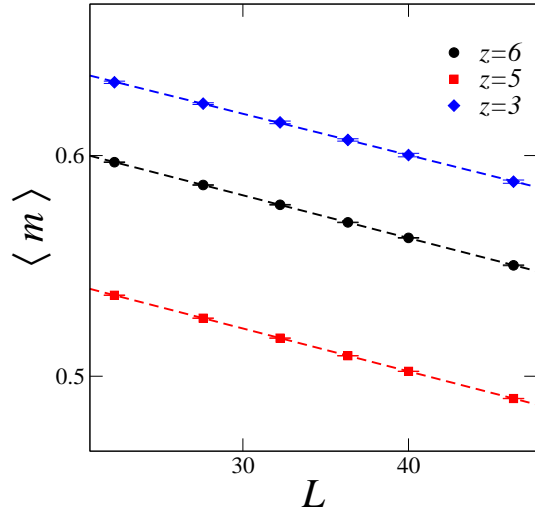


FIG. 5: (Color online) Log-log plot of the order parameter at the critical point for $z = 6$ (black circles), 5 (red squares) and 3 (blue diamonds). The dashed lines are power law fittings.

because the critical point was not well enough evaluated, the uncertainties are ten time bigger than the reported here. This would likely lead to a noticeable shift in the estimates of the critical exponents. Another source of discrepancies is the incorrect definition for the susceptibility used in [11], $\chi = N(\langle m^2 \rangle - \langle m \rangle^2)$, instead of (4). The missing parameter x is necessary in order to have compatibility with the definition for the susceptibility in the Ising model, $\frac{k_B T}{J} \chi = N(\langle m^2 \rangle - \langle m \rangle^2)$, and this may be the cause of the discrepancies in the γ exponent. The same explanation can be applied to the works from Refs. [3, 6, 8, 9]. Finally we must mention that the

TABLE II: Critical values for the MV calculated in this work. For comparison we include the values reported by Santos *et al.* for the MV model and the reported for the Ising model for the bilayer square lattice.

x_c	$1/\nu$	γ/ν	β/ν	$\frac{\gamma+2\beta}{\nu}$	z	model
0.7819(1)	1.03(2)	1.759(7)	0.123(2)	2.005(8)	6	MV
0.7351(1)	1.04(5)	1.756(9)	0.123(3)	2.002(11)	5	MV
0.8721(1)	1.01(2)	1.755(8)	0.123(2)	2.001(9)	3	MV
0.772(10)	0.87(5)	1.59(5)	0.12(4)	1.83(5)	6	MV [11]
0.822(10)	0.87(5)	1.64(5)	0.15(5)	1.96(5)	3	MV [11]
—	1.00(1)	1.750(7)	0.126(7)	2.002(16)	5	Ising [12]

numerical schemes applied in [11] can not be completely trusted, since similar simulations for the Ising model on Archimedean lattices [15] by Lima *et al.* were recently criticized by Malakis *et al.* [16].

Our results show also that the critical point is a monotonic decreasing function on z , with the exception from the bilayer case. In Fig. 6 we can compare the difference in the behavior in the critical point of the MV and the Ising models [12, 17], the critical point for the MV model in square lattice was taken from Ref. [1].

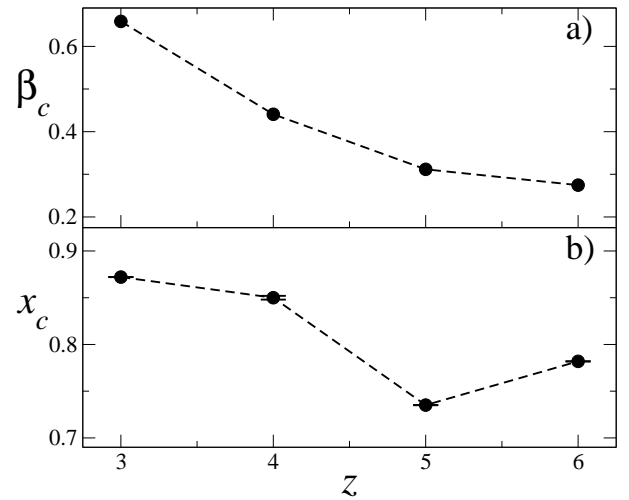


FIG. 6: Critical point behavior for the a) Ising and b) MV models as function of the nearest neighbors number z . We observe that the MV model presents an anomaly at $z = 5$.

The unexpected behavior in the critical point for the bilayer square lattice does not affect the universality class, but further studies are needed in order to check if there is a similar effect of this particular geometry in other non equilibrium systems.

IV. CONCLUSIONS

The MV model on two-dimensional lattices with 3, 5 and 6 nearest neighbors belong to Ising model universality class. Our simulations prove that the set of critical

exponents for both models are consistent in two dimensional regular lattices. The non equilibrium nature of the MV model affects the critical point in the bilayer square lattice, but does not affect the universality class, at least for the static critical exponents. It will be necessary to check if the critical dynamic exponents are the same that those of the Ising model.

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